

On the S-wave pollution of $B \rightarrow K^* l^+ l^-$ observables

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Abstract

It has been argued recently that transverse asymmetries that are expected to be shielded from the presence of the S-wave $K\pi$ pairs originating from the decay of a scalar K_0^* meson, are indeed affected by this pollution due to the impossibility to extract cleanly the normalization for these observables. In this short note we show how using folded distributions, which is nowadays the preferred method to obtain the information from the 4-body decay mode $B \rightarrow K^*(\rightarrow K\pi)l^+l^-$, one can easily bypass this problem and extract the clean observables $P_{1,2,3}$ and also $P'_{4,5,6}$ in a way completely free from this pollution including all lepton mass corrections. We also show that in case one insists in using uniaxial distributions to extract these observables it is possible to reduce this pollution to just lepton mass suppressed terms. On the contrary, the S_i observables, that are by definition normalized by the full differential decay distribution, will indeed suffer from this pollution via their normalization. Finally, we also present a procedure to minimize the error associated to neglecting lepton mass corrections in the distribution defining a massless-improved limit.

The angular distribution $B \rightarrow K^*(\rightarrow K\pi)l^+l^-$ is nowadays, but also in the near future, one of the central players in our search for New Physics in rare B decays. A wide variety of experimental analysis have been presented [1, 2, 3, 4] focusing on different observables related to this rich mode. Simultaneously, there has been a huge effort from the theory community to provide a complete and accurate description of this 4-body decay mode [5]-[16]. Given the relevance of this mode in providing new constraints [17]-[28] it is of utmost importance to control and isolate any possible source of hadronic pollution that can spoil a clear signal of the theory that lies beyond the SM.

In a recent interesting paper [16] one possible source of pollution of the angular distribution $B \rightarrow K^*(\rightarrow K\pi)l^+l^-$ due to events coming from the distribution $B \rightarrow K_0^*(\rightarrow K\pi)l^+l^-$, where K_0^* is a scalar meson resonance, was analyzed. There it was presented the full distribution of the combined channels $B \rightarrow K^*(\rightarrow K\pi)l^+l^-$ and $B \rightarrow K_0^*(\rightarrow K\pi)l^+l^-$ and the focus was on the impact that this pollution could have on the extraction of the transverse asymmetries $A_T^{(2)}$ [6], $A_T^{(\text{re})}$ [14] and $A_T^{(\text{im})}$ [14]. In [16] it was argued that even if one should expect that the transverse asymmetries are unaffected by this S-wave contamination, they are afflicted from this disease via their normalization. They conclude that if the transverse asymmetries (defined with an unusual normalization) are measured in the particular way they propose, they are afflicted by this pollution at the level of less than 10% in all q^2 -range except for $q^2 = 2 \text{ GeV}^2$ where the pollution can be as large as 23%.

In this short note we present a simple procedure to bypass this pollution and extract the transverse asymmetries defined as in [15] in a way completely free from any S-wave contamination including also all lepton mass corrections. First, we show using folded distributions, that an exact ($m_\ell \neq 0$) and clean (no S-wave pollution) extraction of all the P_i observables [15] is perfectly possible. Second, we argue that even using uniangular distributions a strategy can be designed to reduce the impact of the companion decay on the transverse asymmetries to only lepton mass suppressed terms. Incidentally, we also present two procedures to introduce lepton mass corrections inside the observables in the distribution, an exact one preserving the structure of the massless case as much as possible and an approximated one that minimizes the error of neglecting lepton masses. The results presented here aim at reducing substantially two main sources of systematic errors of the experimental data namely the S-wave pollution and lepton mass corrections. The final goal is to provide strategies to enhance the sensitivity to any possible signal of New Physics affecting the clean observables $P_{1,2,3}, P'_{4,5,6}$.

In Section I we recall the definitions of the known observables, introduce new ones and we also present a so called “massless-improved limit”. In Section II we show how to extract the P_i observables in an exact way using folded distributions including all lepton mass corrections with zero S-wave pollution. Also a less clean approximate procedure is discussed using uniangular distributions. In appendices A and B we define the coefficients of the distribution coming from the companion scalar decay $B \rightarrow K_0^*(\rightarrow K\pi)l^+l^-$ and provide more examples of folded distributions.

I. Definitions: Observables, full distribution and massless-improved limit

The angular distribution that describes the four-body decay $B \rightarrow K^*(\rightarrow K\pi)l^+l^-$ including the S-wave pollution from the companion decay $B \rightarrow K_0^*(\rightarrow K\pi)l^+l^-$ is [6, 9, 16]

$$\begin{aligned} \frac{d^4\Gamma}{dq^2 d\cos\theta_K d\cos\theta_l d\phi} = \frac{9}{32\pi} & \left[J_{1s} \sin^2\theta_K + J_{1c} \cos^2\theta_K + (J_{2s} \sin^2\theta_K + J_{2c} \cos^2\theta_K) \cos 2\theta_l \right. \\ & + J_3 \sin^2\theta_K \sin^2\theta_l \cos 2\phi + J_4 \sin 2\theta_K \sin 2\theta_l \cos \phi + J_5 \sin 2\theta_K \sin \theta_l \cos \phi \\ & + (J_{6s} \sin^2\theta_K + J_{6c} \cos^2\theta_K) \cos \theta_l + J_7 \sin 2\theta_K \sin \theta_l \sin \phi + J_8 \sin 2\theta_K \sin 2\theta_l \sin \phi \\ & \left. + J_9 \sin^2\theta_K \sin^2\theta_l \sin 2\phi \right] X + S \end{aligned} \quad (1)$$

where

$$\begin{aligned} S = \frac{1}{4\pi} & \left[\tilde{J}_{1a}^c + \tilde{J}_{1b}^c \cos\theta_K + (\tilde{J}_{2a}^c + \tilde{J}_{2b}^c \cos\theta_K) \cos 2\theta_l + \tilde{J}_4 \sin\theta_K \sin 2\theta_l \cos \phi \right. \\ & \left. + \tilde{J}_5 \sin\theta_K \sin \theta_l \cos \phi + \tilde{J}_7 \sin\theta_K \sin \theta_l \sin \phi + \tilde{J}_8 \sin\theta_K \sin 2\theta_l \sin \phi \right] \end{aligned} \quad (2)$$

and

$$X = \int dm_{K\pi}^2 |BW_{K^*}(m_{K\pi}^2)|^2 \quad (3)$$

being a correction introduced in [16] to take into account the width of the resonance (see [16] for precise definition and details on this function). This correction factorizes from the standard J_i defined in [6, 7, 9] and as, we will show, it will always cancel in the P_i observables. The \tilde{J}_i containing the interference terms coming from the $B \rightarrow K_0^*(\rightarrow K\pi)l^+l^-$ decay computed in [16] are defined in Appendix A.

The full differential decay distribution is then

$$\frac{d\Gamma_{full}}{dq^2} = \frac{d\Gamma_{K^*}}{dq^2} + \frac{d\Gamma_{K_0^*}}{dq^2} \quad (4)$$

with

$$\frac{d\Gamma_{K^*}}{dq^2} = \frac{1}{4} (3J_{1c} + 6J_{1s} - J_{2c} - 2J_{2s}) X, \quad \frac{d\Gamma_{K_0^*}}{dq^2} = +2\tilde{J}_{1a}^c - \frac{2}{3}\tilde{J}_{2a}^c \quad (5)$$

Here we will not consider scalar contributions to facilitate the comparison with [16] where they were also neglected. Notice that the distribution including lepton masses (but neglecting scalars $J_6^c = 0$) contains 11 J_i coefficients (only 10 are independent [10, 15]) plus 8 \tilde{J}_i coefficients coming from the interference with the S-wave polluting process. A generalization of the results presented here to include scalars is straightforward. We take the same inputs as in [28] and all observables should be understood as $J \rightarrow J + \bar{J}$ (obviously the same applies for expressions written in terms of transversity amplitudes).

The first important point is the exact definition of the transverse asymmetries in terms of the coefficients of the distribution. In [16] the transverse asymmetries are defined with an unusual normalization

$$\begin{aligned}\tilde{A}_T^{(2)} &= \frac{4J_3(q^2)}{3J_{1s}(q^2) - J_{2s}(q^2)}, \quad A_T^{(\text{im})}(q^2) = \frac{4J_9(q^2)}{3J_{1s}(q^2) - J_{2s}(q^2)}, \\ A_T^{(\text{re})}(q^2) &= \frac{\beta_\ell J_{6s}(q^2)}{3J_{1s}(q^2) - J_{2s}(q^2)}.\end{aligned}\quad (6)$$

This choice of normalization in [16] is driven by the aim of extracting it from the θ_K uniangular distribution (see below). These definitions albeit correct have two main disadvantages:

- The expression of the transverse asymmetries given in Eq.(6) once expressed in terms of transversity amplitudes involves lepton mass terms via the coefficient¹

$$J_{1s} = \frac{(2 + \beta_\ell^2)}{4} \left[|A_\perp^L|^2 + |A_\parallel^L|^2 + |A_\perp^R|^2 + |A_\parallel^R|^2 \right] + \frac{4m_\ell^2}{q^2} \text{Re} \left(A_\perp^L A_\perp^{R*} + A_\parallel^L A_\parallel^{R*} \right) \quad (7)$$

where $\beta_\ell = \sqrt{1 - 4m_\ell^2/q^2}$. This implies that their definition is not invariant under changes of the lepton mass ($\ell = e, \mu$).

- Normalization requires to measure two coefficients of the distribution.

For these two reasons we prefer to stick to the definition of the P_i observables as given in [15]. These definitions in terms of transversity amplitudes do not depend on any lepton mass term:

$$\begin{aligned}P_1 &= \frac{|A_\perp^L|^2 - |A_\parallel^L|^2 + (L \leftrightarrow R)}{|A_\perp^L|^2 + |A_\parallel^L|^2 + (L \leftrightarrow R)} = \frac{J_3}{2J_{2s}}, \\ P_2 &= \frac{\text{Re}(A_\perp^{L*} A_\parallel^L - A_\perp^R A_\parallel^{R*})}{|A_\perp^L|^2 + |A_\parallel^L|^2 + (L \leftrightarrow R)} = \beta_\ell \frac{J_{6s}}{8J_{2s}}, \\ P_3 &= \frac{\text{Im}(A_\perp^{L*} A_\parallel^L - A_\perp^R A_\parallel^{R*})}{|A_\perp^L|^2 + |A_\parallel^L|^2 + (L \leftrightarrow R)} = -\frac{J_9}{4J_{2s}}\end{aligned}\quad (8)$$

These observables were completed in [28] with

$$\begin{aligned}P'_4 &= \frac{\sqrt{2}\text{Re}(A_0^{L*} A_\parallel^L + A_0^R A_\parallel^{R*})}{\sqrt{\left(|A_\perp^L|^2 + |A_\parallel^L|^2 + (L \leftrightarrow R)\right) \left(|A_0^L|^2 + (L \leftrightarrow R)\right)}} = \frac{J_4}{\sqrt{-J_{2c}J_{2s}}}, \\ P'_5 &= \frac{\sqrt{2}\text{Re}(A_0^{L*} A_\perp^L - A_0^R A_\perp^{R*})}{\sqrt{\left(|A_\perp^L|^2 + |A_\parallel^L|^2 + (L \leftrightarrow R)\right) \left(|A_0^L|^2 + (L \leftrightarrow R)\right)}} = \frac{\beta_\ell}{2} \frac{J_5}{\sqrt{-J_{2c}J_{2s}}}, \\ P'_6 &= \frac{\sqrt{2}\text{Im}(A_0^{L*} A_\parallel^L + A_0^R A_\parallel^{R*})}{\sqrt{\left(|A_\perp^L|^2 + |A_\parallel^L|^2 + (L \leftrightarrow R)\right) \left(|A_0^L|^2 + (L \leftrightarrow R)\right)}} = -\frac{\beta_\ell}{2} \frac{J_7}{\sqrt{-J_{2c}J_{2s}}}\end{aligned}\quad (9)$$

¹From now on we will distinguish between two types of lepton mass dependences: lepton mass terms that will refer to terms like the last term in Eq.(7) and mass dependence via the prefactor β_ℓ .

where the β_ℓ prefactor included in $P_{2,5',6'}$ is there to ensure that the expression in terms of transversity amplitudes does not contain any lepton mass dependence. P_1 is the transverse asymmetry $A_T^{(2)}$ proposed in [6] and P_2 and P_3 correspond to the $A_T^{(\text{re})}$ and $A_T^{(\text{im})}$, respectively, normalized as in [14] and not like in Eq.(6). Notice also that the normalization of the transverse asymmetries in Eqs.(8) requires to measure one single coefficient of the distribution, namely, J_{2s} , free from any lepton mass term.

It is very convenient to express the coefficients of the angular distribution directly in terms of observables. In Ref. [15] it was presented the most complete expression of the coefficients of the distribution including lepton masses and scalars in terms of the full basis of observables. Later on, in [28] a very compact parametrization was found for the massless case without scalar contributions. Here we present a generalization of Ref. [28] to include lepton masses with a similar structure to the massless case at the price of trading the original basis [15]

$$\left\{ \frac{d\Gamma_{K^*}}{dq^2}, A_{\text{FB}}, P_{1,2,3}, P_{4,5,6}, M_1, M_2 \right\} \rightarrow \left\{ \hat{F}_T \frac{d\Gamma_{K^*}}{dq^2}, \hat{F}_L \frac{d\Gamma_{K^*}}{dq^2}, \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2}, \tilde{F}_L \frac{d\Gamma_{K^*}}{dq^2}, P_{1,2,3}, P'_{4,5,6} \right\}$$

The exact parametrization of the coefficients in the massive case is then²

$$\begin{aligned} J_{1s} &= \frac{3}{4} \hat{F}_T \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, & J_{2s} &= \frac{1}{4} \beta_\ell^2 \tilde{F}_T \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, \\ J_{1c} &= \hat{F}_L \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, & J_{2c} &= -\beta_\ell^2 \tilde{F}_L \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, \\ J_3 &= \frac{1}{2} \beta_\ell^2 P_1 \tilde{F}_T \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, & J_{6s} &= 2\beta_\ell P_2 \tilde{F}_T \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, \\ J_4 &= \frac{1}{2} \beta_\ell^2 P'_4 \sqrt{\tilde{F}_T \tilde{F}_L} \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, & J_9 &= -\beta_\ell^2 P_3 \tilde{F}_T \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, \\ J_5 &= \beta_\ell P'_5 \sqrt{\tilde{F}_T \tilde{F}_L} \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, & & \\ J_7 &= -\beta_\ell P'_6 \sqrt{\tilde{F}_T \tilde{F}_L} \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2}, & & \end{aligned} \tag{10}$$

and the redundant but interesting coefficient J_8 is

$$J_8 = -\frac{1}{2} \beta_\ell^2 Q' \sqrt{\tilde{F}_T \tilde{F}_L} \frac{1}{X} \frac{d\Gamma_{K^*}}{dq^2} \tag{11}$$

²Notice that in the first coefficient J_{1s} the β_ℓ dependence does not appear explicitly (but it is included inside \hat{F}_T). The reason is that parametrized in this way and only in this coefficient the mass dependence piece of β_ℓ will help later on to reduce the contribution from the lepton mass term.

where the new observables $\hat{F}_{T,L}$, $\tilde{F}_{T,L}$ are given by

$$\begin{aligned}
\hat{F}_T &= \frac{16}{3N} \left[\frac{(2 + \beta_\ell^2)}{4} \left[|A_\perp^L|^2 + |A_\parallel^L|^2 + |A_\perp^R|^2 + |A_\parallel^R|^2 \right] + \frac{4m_\ell^2}{q^2} \text{Re} \left(A_\perp^L A_\perp^{R*} + A_\parallel^L A_\parallel^{R*} \right) \right] = \frac{16J_{1s}}{3N} \\
\hat{F}_L &= \frac{4}{N} \left[|A_0^L|^2 + |A_0^R|^2 + \frac{4m_\ell^2}{q^2} \left[|A_t|^2 + 2\text{Re}(A_0^L A_0^{R*}) \right] \right] = \frac{4J_{1c}}{N} \\
\tilde{F}_T &= \frac{4}{N} \left[|A_\perp^L|^2 + |A_\parallel^L|^2 + |A_\perp^R|^2 + |A_\parallel^R|^2 \right] = \frac{16J_{2s}}{\beta_\ell^2 N} \\
\tilde{F}_L &= \frac{4}{N} \left[|A_0^L|^2 + |A_0^R|^2 \right] = -\frac{4J_{2c}}{\beta_\ell^2 N}
\end{aligned} \tag{12}$$

and the normalization is

$$\begin{aligned}
N &= \left[(3 + \beta_\ell^2) \left(|A_\perp^L|^2 + |A_\parallel^L|^2 + |A_0^L|^2 + |A_\perp^R|^2 + |A_\parallel^R|^2 + |A_0^R|^2 \right) + \right. \\
&\quad \left. + \frac{12m_\ell^2}{q^2} \left[|A_t|^2 + 2\text{Re}(A_0^L A_0^{R*} + A_\perp^L A_\perp^{R*} + A_\parallel^L A_\parallel^{R*}) \right] \right] = 3J_{1c} + 6J_{1s} - J_{2c} - 2J_{2s} = \frac{4}{X} \frac{d\Gamma_{K^*}}{dq^2}.
\end{aligned}$$

At this point several remarks are in order:

- I. P_i and P'_j q^2 -dependent observables once expressed in terms of transversity amplitudes do not depend on m_ℓ , on the contrary $\hat{F}_{T,L}$ and $\tilde{F}_{T,L}$ do depend.
- II. Eqs.(10) and Eq.(11) are exact concerning lepton masses (no scalars included). In all the folded distributions discussed in next section we will always use this exact parametrization.
- III. They are constructed to resemble Eqs.(14) of [28] for the massless case. The main difference with the massless case is that here there are two longitudinal polarization fractions and two transverse polarization fractions and the sums of transverse and longitudinal fractions differ slightly from one:

$$\hat{F}_L + \hat{F}_T = 1 + \hat{\delta} \quad \tilde{F}_T + \tilde{F}_L = 1 + \delta$$

In the exact massless limit $\hat{F}_L|_{m_\ell=0} = \tilde{F}_L|_{m_\ell=0}$, $\hat{F}_T|_{m_\ell=0} = \tilde{F}_T|_{m_\ell=0}$ and $\hat{\delta}, \delta \rightarrow 0$. A numerical expression for $\delta(s)$ is provided in Table 1. $\hat{\delta}(s)$ will be re-expressed in terms of other quantities below.

- IV. The β_ℓ dependence in the coefficients is kept explicit (except for J_{1s}). This choice will be fundamental when approaching the massless limit to reduce the systematic associated to this limit.

We will explore now in a bit more of detail the massless limit. It is clear that in the exact massive case one should strictly fit the data to four observables $\hat{F}_{L,T}$, $\tilde{F}_{L,T}$ apart from the six clean P_i . The ratios $R_T \equiv \hat{F}_T/\tilde{F}_T$ and $R_L \equiv \hat{F}_L/\tilde{F}_L$ are indeed a direct measurement of the clean observables M_1 and M_2 defined in [15]:

$$R_T = \frac{1}{3} (2 + (1 + 4M_1)\beta_\ell^2) \quad R_L = 1 + M_2$$

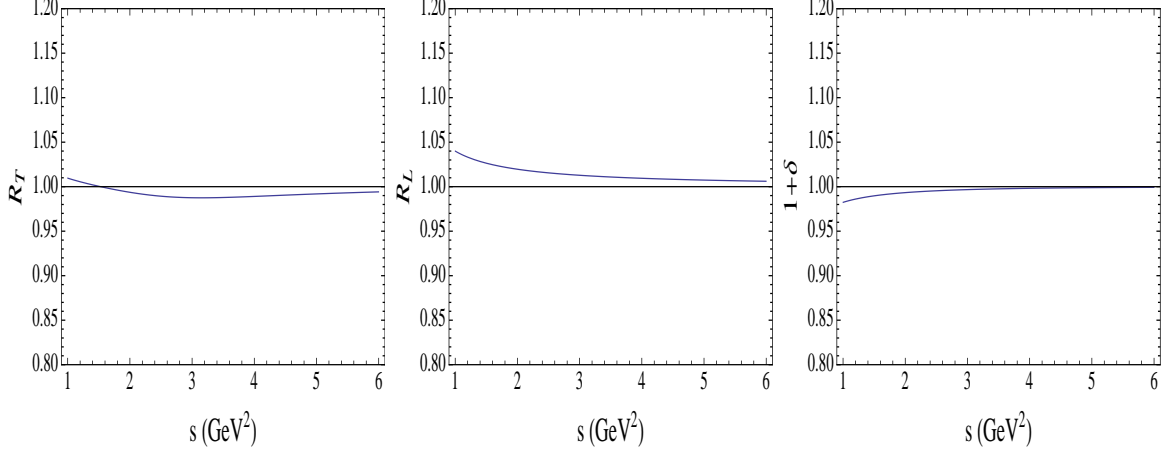


Figure 1: (left) ratio $R_T = \hat{F}_T/\tilde{F}_T$ always below 1.2% in SM, (middle) ratio $R_L = \hat{F}_L/\tilde{F}_L$ with an error range between 1% to less than 4% in the SM (right) δ represents the deviation from one of $\tilde{F}_L + \tilde{F}_T$.

The deviation from one of these ratios in the SM in the range 1 to 6 GeV^2 are shown in Fig.1. They are below 1.2% for R_T and below 4% for R_L . Moreover, given the tight relation of these ratios with $M_{1,2}$ and the relatively small deviations found in these observables in the New Physics scenarios analyzed in [15], a similar range of deviations is expected also for New Physics.

Finally, since it is very costly in statistics to fit for the four observables $\hat{F}_{T,L}, \tilde{F}_{T,L}$ one could take the massless limit ($m_\ell \rightarrow 0$ everywhere) to reduce the number of observables to fit. However, the exact massless limit could induce errors of order 4% (around $q^2 = 1 \text{ GeV}^2$) for instance if $\hat{F}_L = \tilde{F}_L$ is used (see R_L in Fig.1). For this reason we prefer to define an intermediate limit, that we call “massless-improved limit”. This consists in keeping the β_ℓ dependence of Eqs.(10) and reexpressing \hat{F}_T and \tilde{F}_L in terms of \tilde{F}_T in the following way

$$\hat{F}_T = \tilde{F}_T + (R_T - 1)\tilde{F}_T \equiv \tilde{F}_T + L_1 \quad (13)$$

$$\hat{F}_L = 1 - \tilde{F}_T + \delta + (R_L - 1)(1 - \tilde{F}_T) + \mathcal{O}((R_L - 1)\delta) \equiv 1 - \tilde{F}_T + L_2 \quad (14)$$

where $\mathcal{O}((R_L - 1)\delta)$ is much below the permille level and $\hat{\delta}(s) = L_1 + L_2$. A numerical expression for $L_{1,2}$ in the SM can be found in Table 1. In this improved limit the contribution to $\hat{F}_{L,T}$ from lepton mass terms (L_1, L_2) in the SM is always below 1.3% in all q^2 -region from 1 to 6 GeV^2 . Notice that the P_i are not directly affected by this limit since they are always defined by the same $\tilde{F}_{T,L}$, but they are only indirectly affected via the propagation of the error in the determination of $\tilde{F}_{T,L}$. The reason for the sizeable reduction from 4% to near 1.3% in the case of \hat{F}_L is due to the partial cancellation between δ and $(R_L - 1)(1 - \tilde{F}_T)$ in the SM (see Fig.1). This means that the error done by expressing the coefficients of the exact angular distribution Eqs.(10) and Eq.(11) only in terms of $\tilde{F}_{T,L}$ (keeping the β_ℓ dependence but discarding the $L_{1,2}$ corrections) is below 1.3%.

Concerning the integrated observables there are two important observations when including

lepton masses. First notice that

$$\tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2} = \left(\tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2} \right) \Big|_{m_\ell=0}$$

and

$$\sqrt{\tilde{F}_T \tilde{F}_L} \frac{d\Gamma_{K^*}}{dq^2} = \left(\sqrt{\tilde{F}_T \tilde{F}_L} \frac{d\Gamma_{K^*}}{dq^2} \right) \Big|_{m_\ell=0}$$

which means that these products are the same for the massive and massless case. Second important remark is that three of the q^2 -dependent P_i observables include a β_ℓ in their expression in terms of the J_i coefficients and this can be problematic when defining integrated observables. The reason being that what is measured are the J_i coefficients and not products like $\beta_\ell J_i$. Consequently, a small redefinition of three of these observables is required when including lepton masses in the integrated observables. This means that the integrated observables in the massive case naturally split in two categories:

- First set: P_1 , P_3 and P'_4 their q^2 -expression in terms of spin amplitudes and J_i coefficients do not contain neither lepton mass terms nor β_ℓ prefactors.

$$\begin{aligned} \langle P_1^{m_\ell \neq 0} \rangle_{bin} &= \frac{\int_{bin} J_3}{2 \int_{bin} J_{2s}} = \frac{\int_{bin} \beta_\ell^2 P_1 \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2}}{\int_{bin} \beta_\ell^2 \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2}} \\ \langle P_3^{m_\ell \neq 0} \rangle_{bin} &= -\frac{\int_{bin} J_9}{4 \int_{bin} J_{2s}} = \frac{\int_{bin} \beta_\ell^2 P_3 \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2}}{\int_{bin} \beta_\ell^2 \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2}} \\ \langle P'_4{}^{m_\ell \neq 0} \rangle_{bin} &= \frac{\int_{bin} J_4}{\sqrt{-\int_{bin} J_{2c} \int_{bin} J_{2s}}} = \frac{\int_{bin} \beta_\ell^2 P'_4 \sqrt{\tilde{F}_T \tilde{F}_L} \frac{d\Gamma_{K^*}}{dq^2}}{\sqrt{\int_{bin} \beta_\ell^2 \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2} \int_{bin} \beta_\ell^2 \tilde{F}_L \frac{d\Gamma_{K^*}}{dq^2}}}. \end{aligned} \quad (15)$$

For these observables $P_1^{m_\ell \neq 0} = P_1$, $P_3^{m_\ell \neq 0} = P_3$ and $P'_4{}^{m_\ell \neq 0} = P'_4$. Naturally, only when expressing them directly in terms of observables the β_ℓ prefactor arises.

- Second set: P_2 , P'_5 , P'_6 also their q^2 -expression in terms of spin amplitudes do not contain any lepton mass terms or β_ℓ prefactors. However their q^2 -differential expression in terms of J_i **do** contain a β_ℓ prefactor, which means that its corresponding massive definition for the integrated observables requires a redefinition with a prefactor $1/\beta_\ell$:

$$\begin{aligned} \langle P_2^{m_\ell \neq 0} \rangle_{bin} &= \frac{\int_{bin} J_{6s}}{8 \int_{bin} J_{2s}} = \frac{\int_{bin} \beta_\ell^2 (P_2/\beta_\ell) \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2}}{\int_{bin} \beta_\ell^2 \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2}} \\ \langle P'_5{}^{m_\ell \neq 0} \rangle_{bin} &= \frac{\int_{bin} J_5}{2 \sqrt{-\int_{bin} J_{2c} \int_{bin} J_{2s}}} = \frac{\int_{bin} \beta_\ell^2 (P'_5/\beta_\ell) \sqrt{\tilde{F}_T \tilde{F}_L} \frac{d\Gamma_{K^*}}{dq^2}}{\sqrt{\int_{bin} \beta_\ell^2 \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2} \int_{bin} \beta_\ell^2 \tilde{F}_L \frac{d\Gamma_{K^*}}{dq^2}}} \\ \langle P'_6{}^{m_\ell \neq 0} \rangle_{bin} &= -\frac{\int_{bin} J_7}{2 \sqrt{-\int_{bin} J_{2c} \int_{bin} J_{2s}}} = \frac{\int_{bin} \beta_\ell^2 (P'_6/\beta_\ell) \sqrt{\tilde{F}_T \tilde{F}_L} \frac{d\Gamma_{K^*}}{dq^2}}{\sqrt{\int_{bin} \beta_\ell^2 \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2} \int_{bin} \beta_\ell^2 \tilde{F}_L \frac{d\Gamma_{K^*}}{dq^2}}} \end{aligned} \quad (16)$$

where $P_2^{m_\ell \neq 0} = P_2/\beta_\ell$, $P'_5{}^{m_\ell \neq 0} = P'_5/\beta_\ell$ and $P'_6{}^{m_\ell \neq 0} = P'_6/\beta_\ell$.

	1	s	s^2	s^3	s^4	s^5
i	0	1	2	3	4	5
dim	GeV ⁰	GeV ⁻²	GeV ⁻⁴	GeV ⁻⁶	GeV ⁻⁸	GeV ⁻¹⁰
δ_i	-0.0498563	+0.0502874	-0.0228207	+0.00540332	-0.000644906	+0.000030558
$R_{T\,i}$	+1.02522	-0.0074101	-0.0142605	+0.00717305	-0.00120443	+0.0000692328
$R_{L\,i}$	+1.09905	-0.09191	+0.041595	-0.00990168	+0.00118789	-0.0000565414
$L_{1\,i}$	+0.0201856	-0.0273282	+0.0135174	-0.00331905	+0.000403496	-0.0000193199
$L_{2\,i}$	-0.00598948	+0.0255013	-0.0149247	+0.00389817	-0.000488089	+0.0000237576

Table 1: Coefficients of the polynomial approximation to the deviation parameters δ , R_T , R_L , L_1 and L_2 , such that $\delta = \sum_{i=0,5} \delta_i s^i$, $R_T = \sum_{i=0,5} R_{T\,i} s^i$, $R_L = \sum_{i=0,5} R_{L\,i} s^i$, $L_1 = \sum_{i=0,5} L_{1\,i} s^i$ and $L_2 = \sum_{i=0,5} L_{2\,i} s^i$.

The last important point here is to compare the SM predictions for the massive integrated observables defined here $\langle P_i^{m_e \neq 0} \rangle$ with the massless ones computed in [28]. We will focus the comparison on the 1-bin region between 1 to 6 GeV². For the first set due to the balance of β_ℓ^2 factors between numerator and denominator of Eqs.(15) together with the slow variation of this factor with s , one should expect a tiny difference between the massive and the massless prediction. Indeed we found that the massive prediction in the SM is only +0.8% larger than the massless one for P_1 and P'_4 and below permille level for P_3 . In the second set the comparison of the SM integrated observable in the range 1 to 6 GeV² in the massless and massive case is -0.6%, +1.4% and +0.8% for P_2 , P'_5 and P'_6 respectively. This implies that the massless predictions in the SM obtained in [28] are valid to an excellent approximation also in the massive case.

In conclusion in this section we have proposed to use either the exact parametrization or an approximate parametrization so called "massless-improved" limit for the experimental fit (instead of the pure massless limit). This improved parametrization has the advantage of including the dominant contribution from lepton masses with an error when compared to the exact result that amounts only to a maximum of a 1.3% in the SM for the terms that contained $\hat{F}_{L,T}$ in the region between 1 to 6 GeV² while it is exact for the q^2 -dependent P_i observables (up to the propagation of the error in the determination of $\tilde{F}_{T,L}$). The practical consequence of this improved limit is to effectively reduce the basis of observables to one observable less $\{\frac{d\Gamma_{K^*}}{dq^2}, \tilde{F}_T, \tilde{F}_L, P_{1,2,3}, P'_{4,5,6}\}$ (since we are taking $M_1 \rightarrow 0$). If one neglects in addition δ (keeping the β_ℓ dependence) by using $\tilde{F}_L \sim 1 - \tilde{F}_T$ an additional error associated to this second approximation (see Fig.1) should be added. In this case the previous basis gets reduced to eight observables. This step by step procedure will help to know precisely the size of the error associated to each approximation. Once this is done, next step is to compare with the theory predictions. We have shown here that the comparison of the exact massive predictions in the SM with the massless predictions [28] in the 1 to 6 GeV² bin for the integrated observables shows a deviation that in the worst case amounts to 1.4%.

II. Can one extract the observables $P_{1,2,3}$ and $P'_{4,5,6}$ free from S-wave pollution?

In order to answer this question two aspects have to be analyzed. On the one hand, the intrinsic character of the observable with respect to the S-wave pollution and, on the other, the extraction procedure of the observable.

Concerning the first aspect, one should distinguish the observables that does not suffer from S-wave pollution by construction from those that have an inherent structure that includes the pollution. It is clear that the products

$$\tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2}, \quad \tilde{F}_L \frac{d\Gamma_{K^*}}{dq^2}, \quad P_i \tilde{F}_T \frac{d\Gamma_{K^*}}{dq^2} \quad P'_j \sqrt{\tilde{F}_L \tilde{F}_T} \frac{d\Gamma_{K^*}}{dq^2}$$

but also P_i, P'_j with $i = 1, 2, 3$ and $j = 4, 5, 6$ are all intrinsically free from this contamination. Notice that these products enter in the definition of integrated observables. Consequently the corresponding integrated observables are also intrinsically free from S-wave pollution.

On the contrary, for instance, the S_i observables [9] are normalized by the full differential decay distribution

$$S_i = \frac{J_i + \bar{J}_i}{\frac{d\Gamma_{full}}{dq^2}} \quad (17)$$

and they will be directly affected by the S-wave pollution given that the full differential decay distribution is largely affected by this disease (see Eq.(5)). Any other observable normalized by $\frac{d\Gamma_{full}}{dq^2}$, like for example a longitudinal polarization fraction normalized by the full distribution (call it \bar{F}_L), will be strongly S-wave polluted, however the product $\bar{F}_L \frac{d\Gamma_{full}}{dq^2}$ is again free from pollution.

Concerning the extraction procedure, in the following we will show that it is possible to extract these observables completely free from any S-wave pollution in the exact lepton mass case and, afterwards, we will present for completeness also an approximate solution using uniangular distributions.

Exact solution including lepton mass corrections using folded distributions

The crucial point here is to determine the exact procedure to extract the P_i observables. There are basically two choices: uniangular distributions or the full angular distribution. The major drawback of the former is that by integrating angles the distinct angular dependence of the unwanted \tilde{J}_i is lost and mixing (and pollution) with the interesting J_i is unavoidable. If, instead, the full angular distribution is used the major problem is that there is not yet enough statistics to have access to the full distribution. However, the clever idea presented in Ref.[4] of using “folded” distributions, have the double advantage of increasing the statistics and focusing on a restricted set of angular coefficients. This allows to extract information from this mode in a more selective way than traditional uniangular distributions and, in particular, it can be used, as we will show, to isolate the polluting \tilde{J}_i terms. In this note we will exploit this trick to show that using folded distributions one can measure all P_i and P'_j free from any S-wave pollution thanks to their distinct angular dependence.

In Ref. [4], the identification of $\phi \leftrightarrow \phi + \pi$ (when $\phi < 0$) has been used to produce a “folded” angle $\hat{\phi} \in [0, \pi]$ in terms of which a (folded) differential decay rate $d\hat{\Gamma} = d\Gamma(\hat{\phi}) + d\Gamma(\hat{\phi} - \pi)$ is

obtained³. This folding exploits the angular symmetries of the distribution and reduce to a subset the coefficients entering each folded distribution. If we use the same folding procedure including the extra terms coming from the pollution channel $B \rightarrow K_0^*(\rightarrow K\pi)l^+l^-$ one finds the well known structure for the angular distribution together with an extra piece with a distinguishable angular dependence

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos\theta_K d\cos\theta_l d\hat{\phi}} = \frac{9}{16\pi} & \left[J_{1c} \cos^2\theta_K + J_{1s}(1 - \cos^2\theta_K) + J_{2c} \cos^2\theta_K(2\cos^2\theta_\ell - 1) \right. \\ & + J_{2s}(1 - \cos^2\theta_K)(2\cos^2\theta_\ell - 1) + J_3(1 - \cos^2\theta_K)(1 - \cos^2\theta_\ell) \cos 2\hat{\phi} \\ & \left. + J_{6s}(1 - \cos^2\theta_K) \cos\theta_\ell + J_9(1 - \cos^2\theta_K)(1 - \cos^2\theta_\ell) \sin 2\hat{\phi} \right] X + W_1 \quad (18) \end{aligned}$$

where

$$W_1 = \frac{1}{2\pi} \left[\tilde{J}_{1a}^c + \tilde{J}_{1b}^c \cos\theta_K + \left(\tilde{J}_{2a}^c + \tilde{J}_{2b}^c \cos\theta_K \right) (2\cos^2\theta_\ell - 1) \right] \quad (19)$$

The extra factor W_1 concentrates all the S-wave pollution of this folded distribution. Its distinct angular dependence can be parametrized by

$$W_1 = a_W + b_W(2\cos^2\theta_\ell - 1) + c_W \cos\theta_K + d_W \cos\theta_K(2\cos^2\theta_\ell - 1) \quad (20)$$

Fitting for these four parameters (a_W, b_W, c_W, d_W) one can measure and disentangle the S-wave pollution.⁴ But, more importantly, the different angular dependence associated to the relevant coefficients involved in the definition of the $P_{1,2,3}$ permits an extraction of these observables in a way completely free from any S-wave pollution. It is clear from the definition of $P_{1,2,3}$ in Eq.(8) that the folded distribution provides access to J_3 , J_{6s} and J_9 , numerators of P_1 , P_2 and P_3 respectively. But also their common denominator J_{2s} is identified with the coefficient of $(1 - \cos^2\theta_K)(2\cos^2\theta_\ell - 1)$. The normalization of \tilde{A}_T^2 , $A_T^{(\text{re})}$ and $A_T^{(\text{im})}$ presented in [16] requires to measure in addition J_{1c} , unnecessary for the $P_{1,2,3}$.

This is more easily seen writing down Eq.(18) in terms of observables using Eqs.(10). The following expression generalizes Eq.(1) of the LHCb note [4] by including lepton mass corrections and the S-wave pollution in a compact form:

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos\theta_K d\cos\theta_l d\hat{\phi}} = \frac{9}{16\pi} & \left[\hat{F}_L \cos^2\theta_K + \frac{3}{4} \hat{F}_T(1 - \cos^2\theta_K) - \beta_\ell^2 \tilde{F}_L \cos^2\theta_K(2\cos^2\theta_\ell - 1) \right. \\ & + \frac{1}{4} \beta_\ell^2 \tilde{F}_T(1 - \cos^2\theta_K)(2\cos^2\theta_\ell - 1) + \frac{1}{2} \beta_\ell^2 P_1 \tilde{F}_T(1 - \cos^2\theta_K)(1 - \cos^2\theta_\ell) \cos 2\hat{\phi} \\ & \left. + 2\beta_\ell P_2 \tilde{F}_T(1 - \cos^2\theta_K) \cos\theta_\ell - \beta_\ell^2 P_3 \tilde{F}_T(1 - \cos^2\theta_K)(1 - \cos^2\theta_\ell) \sin 2\hat{\phi} \right] \frac{d\Gamma_{K^*}}{dq^2} + W_1 \quad (21) \end{aligned}$$

³Other examples of singled folded distributions exhibiting dependencies on $P_{1,2,3}$ but with the angle θ_K instead can be found in Appendix B.

⁴Notice that if lepton mass suppressed terms from the polluting decay $B \rightarrow K_0^* ll$ are disregarded $a_W = -b_W$ and $c_W = -d_W$

or in the massless-improved limit:

$$\begin{aligned} \frac{d^4 \hat{\Gamma}}{dq^2 d\cos \theta_K d\cos \theta_l d\hat{\phi}} &= \frac{9}{16\pi} \left[(1 - \tilde{F}_T) \cos^2 \theta_K + \frac{3}{4} \tilde{F}_T (1 - \cos^2 \theta_K) - \beta_\ell^2 \tilde{F}_L \cos^2 \theta_K (2 \cos^2 \theta_\ell - 1) \right. \\ &\quad + \frac{1}{4} \beta_\ell^2 \tilde{F}_T (1 - \cos^2 \theta_K) (2 \cos^2 \theta_\ell - 1) + \frac{1}{2} \beta_\ell^2 P_1 \tilde{F}_T (1 - \cos^2 \theta_K) (1 - \cos^2 \theta_\ell) \cos 2\hat{\phi} \\ &\quad \left. + 2\beta_\ell P_2 \tilde{F}_T (1 - \cos^2 \theta_K) \cos \theta_\ell - \beta_\ell^2 P_3 \tilde{F}_T (1 - \cos^2 \theta_K) (1 - \cos^2 \theta_\ell) \sin 2\hat{\phi} \right] \frac{d\Gamma_{K^*}}{dq^2} + W_1 \quad (22) \end{aligned}$$

where the β_ℓ dependence is kept, \hat{F}_L is traded by \tilde{F}_T and the $L_{1,2}$ subleading terms have been dropped off. The error of this improved limit is below 1.3% as discussed in the previous section.

There is also an alternative that can be automatically applied to all the folded distributions discussed here and it is to use $\frac{d\Gamma_{full}}{dq^2}$ instead of $\frac{d\Gamma_{K^*}}{dq^2}$, i.e.,

$$\begin{aligned} \frac{d^4 \hat{\Gamma}}{dq^2 d\cos \theta_K d\cos \theta_l d\hat{\phi}} &= \frac{9}{16\pi} \left[\bar{\tilde{F}}_L \cos^2 \theta_K + \frac{3}{4} \bar{\tilde{F}}_T (1 - \cos^2 \theta_K) - \beta_\ell^2 \bar{\tilde{F}}_L \cos^2 \theta_K (2 \cos^2 \theta_\ell - 1) \right. \\ &\quad + \frac{1}{4} \beta_\ell^2 \bar{\tilde{F}}_T (1 - \cos^2 \theta_K) (2 \cos^2 \theta_\ell - 1) + \frac{1}{2} \beta_\ell^2 P_1 \bar{\tilde{F}}_T (1 - \cos^2 \theta_K) (1 - \cos^2 \theta_\ell) \cos 2\hat{\phi} \\ &\quad \left. + 2\beta_\ell P_2 \bar{\tilde{F}}_T (1 - \cos^2 \theta_K) \cos \theta_\ell - \beta_\ell^2 P_3 \bar{\tilde{F}}_T (1 - \cos^2 \theta_K) (1 - \cos^2 \theta_\ell) \sin 2\hat{\phi} \right] \frac{d\Gamma_{full}}{dq^2} + W_1 \quad (23) \end{aligned}$$

where all quantities with a bar are defined by $\bar{Y} = CY$ with

$$C = \frac{\frac{d\Gamma_{K^*}}{dq^2}}{\frac{d\Gamma_{full}}{dq^2}}$$

Notice that, as explained in the previous section, no S-wave pollution is introduced in this way in the relevant q^2 or integrated P_i observables. Here the improved-massless limit becomes defined by

$$\bar{\tilde{F}}_T = C\hat{F}_T = C\tilde{F}_T + (R_T - 1)C\tilde{F}_T = C\tilde{F}_T + CL_1 \quad (24)$$

where the relative error induced by dropping CL_1 in $C\tilde{F}_T$ is obviously the same as before, below 1.2%. However for $\bar{\tilde{F}}_L$ two possibilities are now open depending on the size of C . First possibility, if $|C - 1|$ is comparable in size to CL_2 taking into account that $C - 1$ is negative while CL_2 is positive a partial cancellation can take place and the best approach is to use

$$\bar{\tilde{F}}_L = C\hat{F}_L = 1 - C\tilde{F}_T + CL_2 + (C - 1) \quad (25)$$

In practice this require $|C - 1|$ to be below 4% to be a good choice (given that L_2 is below 1%). The second possibility is to take

$$\bar{\tilde{F}}_L = C\hat{F}_L = C\tilde{F}_L + (R_L - 1)C\tilde{F}_L \quad (26)$$

In this case the relative error $R_L - 1$ is independent of C exactly as for \tilde{F}_R , however it rises up to 4% at $q^2 = 1 \text{ GeV}^2$ as can be seen in Fig. 1. This choice is safer because the error does not depend on C but the high remaining error (4%) implies that the advantage of the massless-improved limit in front of the massless case is lost.

At this point it is evident the power of this folding technique, in the massive case the full distribution would depend on 11 J_i coefficients (only 10 of them are independent) plus 8 \tilde{J}_i coefficients from the companion decay. Instead, the folded distribution depends only on seven J_i coefficients (or seven observables $\hat{F}_T, \hat{F}_L, \tilde{F}_T, \tilde{F}_L, P_1, P_2, P_3$) and four \tilde{J}_i (a_W, b_W, c_W, d_W). If the improved-massless limit is used the number of observables reduces to six: $d\Gamma_{K^*}/dq^2, \tilde{F}_T, \tilde{F}_L, P_1, P_2, P_3$ plus the four from \tilde{J}_i .

Notice also that the prefactor $X = \int dm_{K\pi}^2 |BW_{K^*}(m_{K\pi}^2)|^2$ cancels out exactly in the P_i given their definition in terms of ratios of coefficients.

One can also exploit further this technique and find several doubled-folded distribution showing different dependencies on the $P_{1,2,3}$ observables and exhibiting common/different S-wave pollutions. For instance, two possible foldings of the angles ϕ and θ_ℓ are

- IV. Identifying $\phi \leftrightarrow \phi + \pi$ when $\phi < 0$ and $\theta_\ell \leftrightarrow \theta_\ell - \frac{\pi}{2}$ when $\theta_\ell > \frac{\pi}{2}$ with $\hat{\phi} \in [0, \pi]$ and $\hat{\theta}_\ell \in [0, \pi/2]$ such that the folded differential distribution is

$$d\hat{\Gamma} = d\Gamma(\hat{\phi}, \hat{\theta}_\ell, \theta_K) + d\Gamma(\hat{\phi}, \hat{\theta}_\ell + \frac{\pi}{2}, \theta_K) + d\Gamma(\hat{\phi} - \pi, \hat{\theta}_\ell, \theta_K) + d\Gamma(\hat{\phi} - \pi, \hat{\theta}_\ell + \frac{\pi}{2}, \theta_K)$$

corresponding to

$$\begin{aligned} \frac{d^4 \hat{\Gamma}}{dq^2 d\cos \theta_K d\cos \hat{\theta}_\ell d\hat{\phi}} &= \frac{9}{32\pi} \left[4\hat{F}_L \cos^2 \theta_K + 3\hat{F}_T \sin^2 \theta_K + \tilde{F}_T \sin^2 \theta_K (\beta_\ell^2 P_1 \cos 2\hat{\phi} \right. \\ &\quad \left. + 4\beta_\ell P_2 (\cos \hat{\theta}_\ell - \sin \hat{\theta}_\ell) - 2\beta_\ell^2 P_3 \sin 2\hat{\phi}) \right] \frac{d\Gamma_{K^*}}{dq^2} + W_4 \end{aligned} \quad (27)$$

where

$$W_4 = \frac{1}{\pi} \left[\tilde{J}_{1a}^c + \tilde{J}_{1b}^c \cos \theta_K \right]$$

- V. Identifying $\phi \leftrightarrow \phi + \pi$ when $\phi < 0$ and $\theta_\ell \leftrightarrow \pi - \theta_\ell$ when $\theta_\ell > \frac{\pi}{2}$ with $\hat{\phi} \in [0, \pi]$ and $\hat{\theta}_\ell \in [0, \pi/2]$ and the folded distribution

$$d\hat{\Gamma} = d\Gamma(\hat{\phi}, \hat{\theta}_\ell, \theta_K) + d\Gamma(\hat{\phi}, \pi - \hat{\theta}_\ell, \theta_K) + d\Gamma(\hat{\phi} - \pi, \hat{\theta}_\ell, \theta_K) + d\Gamma(\hat{\phi} - \pi, \pi - \hat{\theta}_\ell, \theta_K)$$

corresponding to

$$\begin{aligned} \frac{d^4 \hat{\Gamma}}{dq^2 d\cos \theta_K d\cos \hat{\theta}_\ell d\hat{\phi}} &= \frac{9}{32\pi} \left[4\cos^2 \theta_K (\hat{F}_L - \beta_\ell^2 \tilde{F}_L \cos 2\hat{\theta}_\ell) + \sin^2 \theta_K (3\hat{F}_T + \beta_\ell^2 \tilde{F}_T \cos 2\hat{\theta}_\ell) + \right. \\ &\quad \left. + 2\beta_\ell^2 \sin^2 \theta_K \tilde{F}_T (P_1 \cos 2\hat{\phi} - 2P_3 \sin 2\hat{\phi}) \sin^2 \hat{\theta}_\ell \right] \frac{d\Gamma_{K^*}}{dq^2} + W_5 \end{aligned} \quad (28)$$

where $W_5 = 2W_1(\hat{\phi}, \hat{\theta}_\ell, \theta_K)$.

P_1	P_2	P_3	P'_4	P'_5	P'_6
(1)-(13)	(1)-(4), (6)-(10),(12),(13)	(1)-(7),(11)-(12)	(10)	(10)-(13)	(11),(12)

Table 2: Sensitivities to each P_i observable from the set of 13 folding distributions discussed.

Other double foldings with the angles ϕ and θ_K that offer a different dependence on the observables $P_{1,2,3}$ can be found in Appendix B.

Finally, we show how the pollution can be also disentangled from the $P'_{4,5,6}$ observables using this folding technique. Here we present different possibilities:

- X. Identifying the angle $\phi \leftrightarrow -\phi$ when $\phi < 0$ with $\hat{\phi} \in [0, \pi]$ and the differential folded distribution is now $d\hat{\Gamma} = d\Gamma(\hat{\phi}, \theta_\ell, \theta_K) + d\Gamma(-\hat{\phi}, \theta_\ell, \theta_K)$ where

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos\theta_K d\cos\theta_\ell d\hat{\phi}} &= \frac{9}{64\pi} \left[4\cos^2\theta_K(\hat{F}_L - \beta_\ell^2 \tilde{F}_L \cos 2\theta_\ell) + (3\hat{F}_T + \beta_\ell^2 \tilde{F}_T \cos 2\theta_\ell) \sin^2\theta_K + \right. \\ &\quad + 2\tilde{F}_T(\beta_\ell^2 P_1 \cos 2\hat{\phi} \sin^2\theta_K \sin^2\theta_\ell + 4\beta_\ell P_2 \cos\theta_\ell \sin^2\theta_K) + \\ &\quad \left. + 2\sqrt{\tilde{F}_L \tilde{F}_T}(\beta_\ell^2 P'_4 \sin 2\theta_K \sin 2\theta_\ell + 2\beta_\ell P'_5 \sin 2\theta_K \sin\theta_\ell) \cos\hat{\phi} \right] \frac{d\Gamma_{K*}}{dq^2} + W_{10} \quad (29) \end{aligned}$$

with

$$W_{10} = \frac{1}{2\pi} \left[\tilde{J}_{1a}^c + \tilde{J}_{2a}^c \cos 2\theta_\ell + \cos\theta_K(\tilde{J}_{1b}^c + \tilde{J}_{2b}^c \cos 2\theta_\ell) + \cos\hat{\phi}(\tilde{J}_5 + 2\tilde{J}_4 \cos\theta_\ell) \sin\theta_K \sin\theta_\ell \right]$$

- XI. Identifying the angle $\theta_\ell \leftrightarrow \pi - \theta_\ell$ when $\theta_\ell > \frac{\pi}{2}$ with $\hat{\theta}_\ell \in [0, \pi/2]$ and the folded distribution $d\hat{\Gamma} = d\Gamma(\phi, \hat{\theta}_\ell, \theta_K) + d\Gamma(\phi, \pi - \hat{\theta}_\ell, \theta_K)$ is then

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos\theta_K d\cos\hat{\theta}_\ell d\phi} &= \frac{9}{64\pi} \left[4\cos^2\theta_K(\hat{F}_L - \beta_\ell^2 \tilde{F}_L \cos 2\hat{\theta}_\ell) + (3\hat{F}_T + \beta_\ell^2 \tilde{F}_T \cos 2\hat{\theta}_\ell) \sin^2\theta_K \right. \\ &\quad + 2\beta_\ell^2 \tilde{F}_T(P_1 \cos 2\phi \sin^2\theta_K \sin^2\hat{\theta}_\ell - 2P_3 \sin 2\phi \sin^2\hat{\theta}_\ell \sin^2\theta_K) + \\ &\quad \left. + 4\beta_\ell \sqrt{\tilde{F}_L \tilde{F}_T}(P'_5 \cos\phi - P'_6 \sin\phi) \sin 2\theta_K \sin\hat{\theta}_\ell \right] \frac{d\Gamma_{K*}}{dq^2} + W_{11} \end{aligned}$$

where

$$W_{11} = \frac{1}{2\pi} \left[\tilde{J}_{1a}^c + \tilde{J}_{2a}^c \cos 2\hat{\theta}_\ell + \cos\theta_K(\tilde{J}_{1b}^c + \tilde{J}_{2b}^c \cos 2\hat{\theta}_\ell) + (\tilde{J}_5 \cos\phi + \tilde{J}_7 \sin\phi) \sin\theta_K \sin\hat{\theta}_\ell \right]$$

- XII. Identifying $\theta_\ell \leftrightarrow \theta_\ell - \frac{\pi}{2}$ when $\theta_\ell > \frac{\pi}{2}$ with $\hat{\theta}_\ell \in [0, \pi/2]$ and the folded distribution $d\hat{\Gamma} = d\Gamma(\phi, \hat{\theta}_\ell, \theta_K) + d\Gamma(\phi, \hat{\theta}_\ell + \frac{\pi}{2}, \theta_K)$ is

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos\theta_K d\cos\hat{\theta}_\ell d\phi} &= \frac{9}{32\pi} \left[\frac{1}{4}(4\hat{F}_L + 3\hat{F}_T + (4\hat{F}_L - 3\hat{F}_T) \cos 2\theta_K) + \right. \\ &\quad + \tilde{F}_T(\frac{1}{2}\beta_\ell^2 P_1 \cos 2\phi \sin^2\theta_K + 2\beta_\ell P_2 \sin^2\theta_K(\cos\hat{\theta}_\ell - \sin\hat{\theta}_\ell) - \beta_\ell^2 P_3 \sin 2\phi \sin^2\theta_K) \\ &\quad \left. + \beta_\ell \sqrt{\tilde{F}_L \tilde{F}_T}(P'_5 \cos\phi - P'_6 \sin\phi) \sin 2\theta_K(\sin\hat{\theta}_\ell + \cos\hat{\theta}_\ell) \right] \frac{d\Gamma_{K*}}{dq^2} + W_{12} \end{aligned}$$

where

$$W_{12} = \frac{1}{4\pi} \left[2\tilde{J}_{1a}^c + 2\tilde{J}_{1b}^c \cos \theta_K + (\tilde{J}_5 \cos \phi + \tilde{J}_7 \sin \phi) \sin \theta_K (\cos \hat{\theta}_\ell + \sin \hat{\theta}_\ell) \right]$$

One last example of two-folded angle distribution sensitive to P_5' comes from identifying $\phi \leftrightarrow -\phi$ when $\phi < 0$ and $\theta_\ell \leftrightarrow \theta_\ell - \frac{\pi}{2}$ when $\theta_\ell > \frac{\pi}{2}$. The corresponding combination is

$$d\hat{\Gamma} = d\Gamma(\hat{\phi}, \hat{\theta}_\ell, \theta_K) + d\Gamma(\hat{\phi}, \hat{\theta}_\ell + \frac{\pi}{2}, \theta_K) + d\Gamma(-\hat{\phi}, \hat{\theta}_\ell, \theta_K) + d\Gamma(-\hat{\phi}, \hat{\theta}_\ell + \frac{\pi}{2}, \theta_K)$$

where

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos \theta_K d\cos \hat{\theta}_\ell d\hat{\phi}} = & \frac{9}{32\pi} \left[\frac{1}{2} (4\hat{F}_L + 3\hat{F}_T + (4\hat{F}_L - 3\hat{F}_T) \cos 2\theta_K) + \right. \\ & + 2\beta_\ell \sqrt{\tilde{F}_L \tilde{F}_T} P_5' \cos \hat{\phi} \sin 2\theta_K (\sin \hat{\theta}_\ell + \cos \hat{\theta}_\ell) + \\ & \left. + \tilde{F}_T \sin^2 \theta_K (\beta_\ell^2 P_1 \cos 2\hat{\phi} + 4\beta_\ell P_2 (\cos \hat{\theta}_\ell - \sin \hat{\theta}_\ell)) \right] \frac{d\Gamma_{K^*}}{dq^2} + W_{13} \end{aligned}$$

with

$$W_{13} = \frac{1}{2\pi} \left[2\tilde{J}_{1a}^c + 2\tilde{J}_{1b}^c \cos \theta_K + \tilde{J}_5 \cos \hat{\phi} \sin \theta_K (\cos \hat{\theta}_\ell + \sin \hat{\theta}_\ell) \right]$$

Of course, using the same procedure also the integrated observables can be extracted cleanly from the folded distributions in a way completely free from any S-wave pollution.

Approximate solution using uniangular distributions

In Ref[16] the 'unusual' normalization factor $3J_{1s} - J_{2s}$ of the transverse asymmetries is obtained from the uniangular distribution

$$\frac{d^2\Gamma}{dq^2 d\cos \theta_K} = a_{\theta_K}(q^2) + b_{\theta_K}(q^2) \cos \theta_K + c_{\theta_K}(q^2) \cos^2 \theta_K, \quad (30)$$

where the coefficient functions are easily obtained integrating the angular distribution Eq.(1) over the angles ϕ and θ_ℓ . In particular, in our notation one finds

$$\begin{aligned} a_{\theta_K}(q^2) &= \tilde{J}_{1a}^c - \frac{1}{3} \tilde{J}_{2a}^c + \frac{3}{8} (3J_{1s} - J_{2s}) X = \tilde{J}_{1a}^c - \frac{1}{3} \tilde{J}_{2a}^c + \frac{3}{32} (9\hat{F}_T - \beta^2 \tilde{F}_T) \frac{d\Gamma_{K^*}}{dq^2} \\ &\equiv \frac{3}{32} (9\hat{F}_T - \beta^2 \tilde{F}_T) (1 + \Delta(q^2)) \frac{d\Gamma_{K^*}}{dq^2} \end{aligned} \quad (31)$$

which corresponds exactly to Eq.(26) of Ref.[16] and defines in our notation the function $\Delta(q^2)$ introduced in Ref.[16]. This function $\Delta(q^2)$ is the responsible for the deviation up to 23% around $q^2 = 2 \text{ GeV}^2$. Indeed it is not surprising this huge effect. It is immediate to see from the definition of $\Delta(q^2)$ in [16] in terms of transversity amplitudes that it is proportional to $|\mathcal{M}_0'^L|^2 + |\mathcal{M}_0'^R|^2$ (see Ref.[16] for definitions) which is the source of this pollution and large deviation. Notice that with the procedure described in [16] this pollution is not suppressed by any m_ℓ^2/q^2 prefactor.

However, if one uses instead the observables P_1, P_2 and P_3 defined as in Eq.(8) one can design a procedure to extract their normalization J_{2s} using uniangular distributions with a lepton mass suppressed S-wave pollution. In the region between 1 to 6 GeV² this amounts to a much small pollution than the one found using $\Delta(q^2)$ coming from unsuppressed lepton mass terms.

We should emphasize that using uniangular distributions is not the best choice, since as we have shown folded distributions allow for an exact extraction of the P_i free from any pollution. Still if one insist in using uniangular distributions a procedure to obtain this normalization is easily constructed using the distributions on θ_ℓ and ϕ . The idea is to combine the two uniangular distributions

$$\frac{d^2\Gamma}{dq^2 d\cos\theta_\ell} = a_{\theta_\ell} + b_{\theta_\ell} \cos\theta_\ell + c_{\theta_\ell} \cos^2\theta_\ell \quad (32)$$

$$\frac{d^2\Gamma}{dq^2 d\phi} = a_\phi + b_\phi \cos\phi + c_\phi \sin\phi + d_\phi \cos 2\phi + e_\phi \sin 2\phi \quad (33)$$

where the coefficients can be trivially obtained by integrating the corresponding angles. Two possible combinations relevant for our purposes are:

$$\begin{aligned} -a_{\theta_\ell} + 3\pi \frac{1 + \beta_\ell^2}{3 + \beta_\ell^2} a_\phi &= K_1 \\ a_{\theta_\ell} + \frac{1 + \beta_\ell^2}{2\beta_\ell^2} c_{\theta_\ell} &= K_2 \end{aligned} \quad (34)$$

where

$$\begin{aligned} K_1 &= \frac{1}{8(3 + \beta_\ell^2)} \left[3\beta_\ell^2 \left(3\hat{F}_T + \tilde{F}_T + 2(\hat{F}_L - \tilde{F}_L) \right) \frac{d\Gamma_{K^*}}{dq^2} + 16(\beta_\ell^2 \tilde{J}_{1a}^c + \tilde{J}_{2a}^c) \right] \\ K_2 &= \frac{3 + \beta_\ell^2}{2\beta_\ell^2} K_1 \end{aligned} \quad (35)$$

In turn they can be written as

$$K_1 = \frac{3}{2} X J_{2s} + \mathcal{O}\left(\frac{m_\ell^2}{q^2}\right) \quad K_2 = 3X J_{2s} + \mathcal{O}\left(\frac{m_\ell^2}{q^2}\right)$$

which shows that both combinations allow the extraction of the normalization coefficient J_{2s} with an error given by terms of order $\mathcal{O}(m_\ell^2/q^2)$.

The influence of the $\mathcal{O}(m_\ell^2/q^2)$ term is moderate in the region of interest between 1 and 6 GeV² (below 1 GeV² more difficult problems and uncertainties arises).

Conclusions

We have shown that the P_i observables are not only intrinsically clean observables from S-wave pollution point of view, but also that they can be extracted using folded distributions in a way completely free from any S-wave contamination coming from the companion decay $B \rightarrow K_0^{*l^+l^-}$, including all lepton mass corrections. The same conclusion applies to the corresponding integrated observables.

We have also defined a massless-improved limit for the differential distribution, an intermediate stage between the massive and the massless case, that effectively reduces the number of observables at a relatively low cost: an error below 1.3% in the SM in the region between 1 to 6 GeV². Also a comparison of the SM prediction for the integrated observables $\langle P_i \rangle_{q^2=1-6\text{GeV}^2}$ in the massive and massless case shows a discrepancy below 1.4% in the SM, implying that the SM massless predictions for the integrated observables given in [28] are an excellent approximation.

Both results are important experimentally to get a better control or even remove the systematics associated to S-wave pollution and lepton masses.

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Appendix A. \tilde{J}_i definitions

The definition of the \tilde{J}_i derive directly from the full angular distribution given in [16]

$$\begin{aligned}
\tilde{J}_{1a}^c &= \mathcal{I}_1^{c'}(q^2) \int |BW_{K_0^*}(m_{K\pi}^2)|^2 dm_{K\pi}^2 \\
\tilde{J}_{2a}^c &= \mathcal{I}_2^{c'}(q^2) \int |BW_{K_0^*}(m_{K\pi}^2)|^2 dm_{K\pi}^2 \\
\tilde{J}_{1b}^c &= 2\sqrt{3} \int \text{Re} \left[\mathcal{I}_1^{c''} BW_{K_0^*}(m_{K\pi}^2) BW_{K^*}^\dagger(m_{K\pi}^2) \right] dm_{K\pi}^2 \\
\tilde{J}_{2b}^c &= 2\sqrt{3} \int \text{Re} \left[\mathcal{I}_2^{c''}(q^2) BW_{K_0^*}(m_{K\pi}^2) BW_{K^*}^\dagger(m_{K\pi}^2) \right] dm_{K\pi}^2 \\
\tilde{J}_4 &= 2\sqrt{3} \int \text{Re} \left[\mathcal{I}_4''(q^2) BW_{K_0^*}(m_{K\pi}^2) BW_{K^*}^\dagger(m_{K\pi}^2) \right] dm_{K\pi}^2 \\
\tilde{J}_5 &= 2\sqrt{3} \int \text{Re} \left[\mathcal{I}_5''(q^2) BW_{K_0^*}(m_{K\pi}^2) BW_{K^*}^\dagger(m_{K\pi}^2) \right] dm_{K\pi}^2 \\
\tilde{J}_7 &= 2\sqrt{3} \int \text{Im} \left[\mathcal{I}_7''(q^2) BW_{K_0^*}(m_{K\pi}^2) BW_{K^*}^\dagger(m_{K\pi}^2) \right] dm_{K\pi}^2 \\
\tilde{J}_8 &= 2\sqrt{3} \int \text{Im} \left[\mathcal{I}_8''(q^2) BW_{K_0^*}(m_{K\pi}^2) BW_{K^*}^\dagger(m_{K\pi}^2) \right] dm_{K\pi}^2
\end{aligned} \tag{36}$$

We refer the reader to [16] for definitions of \mathcal{I}_i and Breight-Wigner resonances $BW_{K_0^*}(m_{K\pi}^2)$, $BW_{K^*}(m_{K\pi}^2)$. See also [29].

Appendix B. More folded distributions

Observables $P_{1,2,3}$

- Single folded distributions: The angle θ_K can be folded in two different ways.

II. The first comes from the identification $\theta_K \leftrightarrow \pi - \theta_K$ when $\theta_K > \frac{\pi}{2}$ with $\hat{\theta}_K \in [0, \pi/2]$ implying the combination $d\hat{\Gamma} = d\Gamma(\phi, \theta_\ell, \hat{\theta}_K) + d\Gamma(\phi, \theta_\ell, \pi - \hat{\theta}_K)$ corresponding to

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos\hat{\theta}_K d\cos\theta_\ell d\phi} &= \frac{9}{64\pi} \left[4\cos^2\hat{\theta}_K(\hat{F}_L - \beta_\ell^2\tilde{F}_L\cos 2\theta_\ell) + \sin^2\hat{\theta}_K(3\hat{F}_T + \beta_\ell^2\tilde{F}_T\cos 2\theta_\ell) + \right. \\ &\quad \left. + 2\tilde{F}_T(\beta_\ell^2 P_1 \cos 2\phi \sin^2\hat{\theta}_K \sin^2\theta_\ell + 4\beta_\ell P_2 \cos\theta_\ell \sin^2\hat{\theta}_K \right. \\ &\quad \left. - 2\beta_\ell^2 P_3 \sin 2\phi \sin^2\hat{\theta}_K \sin^2\theta_\ell) \right] \frac{d\Gamma_{K*}}{dq^2} + W_2 \end{aligned} \quad (37)$$

where

$$W_2 = \frac{1}{2\pi} \left[\tilde{J}_{1a}^c + \tilde{J}_{2a}^c \cos 2\theta_\ell + (\cos\phi(\tilde{J}_5 + 2\tilde{J}_4 \cos\theta_\ell) + (\tilde{J}_7 + 2\tilde{J}_8 \cos\theta_\ell) \sin\phi) \sin\hat{\theta}_K \sin\theta_\ell \right]$$

III. The second is $\theta_K \leftrightarrow \theta_K - \frac{\pi}{2}$ when $\theta_K > \frac{\pi}{2}$ with $\hat{\theta}_K \in [0, \pi/2]$ and the combination is $d\hat{\Gamma} = d\Gamma(\phi, \theta_\ell, \theta_K) + d\Gamma(\phi, \theta_\ell, \theta_K + \frac{\pi}{2})$ given by

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos\hat{\theta}_K d\cos\theta_\ell d\phi} &= \frac{9}{128\pi} \left[4\hat{F}_L + 3\hat{F}_T + (-4\tilde{F}_L + \tilde{F}_T)\beta_\ell^2 \cos 2\theta_\ell + \right. \\ &\quad \left. + 2\tilde{F}_T(\beta_\ell^2 P_1 \cos 2\phi \sin^2\theta_\ell + 4\beta_\ell P_2 \cos\theta_\ell - 2\beta_\ell^2 P_3 \sin 2\phi \sin^2\theta_\ell) \right] \frac{d\Gamma_{K*}}{dq^2} + W_3 \end{aligned} \quad (38)$$

where

$$\begin{aligned} W_3 &= \frac{1}{4\pi} \left[2(\tilde{J}_{1a}^c + \tilde{J}_{2a}^c \cos 2\theta_\ell) + (\tilde{J}_{1b}^c + \tilde{J}_{2b}^c \cos 2\theta_\ell)(\cos\hat{\theta}_K - \sin\hat{\theta}_K) + (\cos\phi(\tilde{J}_5 + 2\tilde{J}_4 \cos\theta_\ell) \right. \\ &\quad \left. + (\tilde{J}_7 + 2\tilde{J}_8 \cos\theta_\ell) \sin\phi)(\cos\hat{\theta}_K + \sin\hat{\theta}_K) \sin\theta_\ell \right] \end{aligned} \quad (39)$$

- Double folded distributions:

VI. Identifying $\phi \leftrightarrow \phi + \pi$ when $\phi < 0$ and $\theta_K \leftrightarrow \theta_K - \frac{\pi}{2}$ when $\theta_K > \frac{\pi}{2}$ with $\hat{\phi} \in [0, \pi]$ and $\hat{\theta}_K \in [0, \pi/2]$ with a combined distribution

$$d\hat{\Gamma} = d\Gamma(\hat{\phi}, \theta_\ell, \hat{\theta}_K) + d\Gamma(\hat{\phi}, \theta_\ell, \hat{\theta}_K + \frac{\pi}{2}) + d\Gamma(\hat{\phi} - \pi, \theta_\ell, \hat{\theta}_K) + d\Gamma(\hat{\phi} - \pi, \theta_\ell, \hat{\theta}_K + \frac{\pi}{2})$$

given by

$$\begin{aligned} \frac{d^4\hat{\Gamma}}{dq^2 d\cos\hat{\theta}_K d\cos\theta_\ell d\hat{\phi}} &= \frac{9}{64\pi} \left[4\hat{F}_L + 3\hat{F}_T + (-4\tilde{F}_L + \tilde{F}_T)\beta_\ell^2 \cos 2\theta_\ell + 8\tilde{F}_T\beta_\ell P_2 \cos\theta_\ell + \right. \\ &\quad \left. + 2\beta_\ell^2\tilde{F}_T(P_1 \cos 2\hat{\phi} - 2P_3 \sin 2\hat{\phi}) \sin^2\theta_\ell \right] \frac{d\Gamma_{K*}}{dq^2} + W_6 \end{aligned} \quad (40)$$

where

$$W_6 = \frac{1}{2\pi} \left[2(\tilde{J}_{1a}^c + \tilde{J}_{2a}^c \cos 2\theta_\ell) + (\tilde{J}_{1b}^c + \tilde{J}_{2b}^c \cos 2\theta_\ell)(\cos \hat{\theta}_K - \sin \hat{\theta}_K) \right]$$

VII. Identifying $\phi \leftrightarrow \phi + \pi$ when $\phi < 0$ and $\theta_K \leftrightarrow \pi - \theta_K$ when $\theta_K > \frac{\pi}{2}$ with $\hat{\phi} \in [0, \pi]$ and $\hat{\theta}_K \in [0, \pi/2]$ and the combined distribution

$$d\hat{\Gamma} = d\Gamma(\hat{\phi}, \theta_\ell, \hat{\theta}_K) + d\Gamma(\hat{\phi}, \theta_\ell, \pi - \hat{\theta}_K) + d\Gamma(\hat{\phi} - \pi, \theta_\ell, \hat{\theta}_K) + d\Gamma(\hat{\phi} - \pi, \theta_\ell, \pi - \hat{\theta}_K)$$

where

$$\begin{aligned} \frac{d^4 \hat{\Gamma}}{dq^2 d\cos \hat{\theta}_K d\cos \theta_\ell d\hat{\phi}} &= \frac{9}{32\pi} \left[4\cos^2 \hat{\theta}_K (\hat{F}_L - \beta_\ell^2 \tilde{F}_L \cos 2\theta_\ell) + \sin^2 \hat{\theta}_K (3\hat{F}_T + \beta_\ell^2 \tilde{F}_T \cos 2\theta_\ell) + \right. \\ &\quad \left. + 2\sin^2 \hat{\theta}_K \tilde{F}_T \left(\beta_\ell^2 (P_1 \cos 2\hat{\phi} - 2P_3 \sin 2\hat{\phi}) \sin^2 \theta_\ell + 4\beta_\ell P_2 \cos \theta_\ell \right) \right] \frac{d\Gamma_{K*}}{dq^2} + W_7 \end{aligned} \quad (41)$$

with

$$W_7 = \frac{1}{\pi} \left[\tilde{J}_{1a}^c + \tilde{J}_{2a}^c \cos 2\theta_\ell \right]$$

VIII. Identifying $\phi \leftrightarrow -\phi$ when $\phi < 0$ and $\theta_K \leftrightarrow \pi - \theta_K$ when $\theta_K > \frac{\pi}{2}$ with $\hat{\phi} \in [0, \pi]$ and $\hat{\theta}_K \in [0, \pi/2]$ with the combined distribution

$$d\hat{\Gamma} = d\Gamma(\hat{\phi}, \theta_\ell, \hat{\theta}_K) + d\Gamma(\hat{\phi}, \theta_\ell, \pi - \hat{\theta}_K) + d\Gamma(-\hat{\phi}, \theta_\ell, \hat{\theta}_K) + d\Gamma(-\hat{\phi}, \theta_\ell, \pi - \hat{\theta}_K)$$

where

$$\begin{aligned} \frac{d^4 \hat{\Gamma}}{dq^2 d\cos \hat{\theta}_K d\cos \theta_\ell d\hat{\phi}} &= \frac{9}{32\pi} \left[4\cos^2 \hat{\theta}_K (\hat{F}_L - \beta_\ell^2 \tilde{F}_L \cos 2\theta_\ell) + (3\hat{F}_T + \beta_\ell^2 \tilde{F}_T \cos 2\theta_\ell) \sin^2 \hat{\theta}_K \right. \\ &\quad \left. + 2\tilde{F}_T (\beta_\ell^2 P_1 \cos 2\hat{\phi} \sin^2 \hat{\theta}_K \sin^2 \theta_\ell + 4\beta_\ell P_2 \cos \theta_\ell \sin^2 \hat{\theta}_K) \right] \frac{d\Gamma_{K*}}{dq^2} + W_8 \end{aligned} \quad (42)$$

with

$$W_8 = \frac{1}{\pi} \left[\tilde{J}_{1a}^c + \tilde{J}_{2a}^c \cos 2\theta_\ell + \cos \hat{\phi} (\tilde{J}_5 + 2\tilde{J}_4 \cos \theta_\ell) \sin \hat{\theta}_K \sin \theta_\ell \right]$$

IX. Identifying $\phi \leftrightarrow -\phi$ when $\phi < 0$ and $\theta_K \leftrightarrow \theta_K - \frac{\pi}{2}$ when $\theta_K > \frac{\pi}{2}$ with $\hat{\phi} \in [0, \pi]$ and $\hat{\theta}_K \in [0, \pi/2]$ with the combined distribution

$$d\hat{\Gamma} = d\Gamma(\hat{\phi}, \theta_\ell, \hat{\theta}_K) + d\Gamma(\hat{\phi}, \theta_\ell, \hat{\theta}_K + \frac{\pi}{2}) + d\Gamma(-\hat{\phi}, \theta_\ell, \hat{\theta}_K) + d\Gamma(-\hat{\phi}, \theta_\ell, \hat{\theta}_K + \frac{\pi}{2})$$

where

$$\begin{aligned} \frac{d^4 \hat{\Gamma}}{dq^2 d\cos \hat{\theta}_K d\cos \theta_\ell d\hat{\phi}} &= \frac{9}{64\pi} \left[4\hat{F}_L + 3\hat{F}_T + \beta_\ell^2 (-4\tilde{F}_L + \tilde{F}_T) \cos 2\theta_\ell \right. \\ &\quad \left. + 2\tilde{F}_T (\beta_\ell^2 P_1 \cos 2\hat{\phi} \sin^2 \theta_\ell + 4\beta_\ell P_2 \cos \theta_\ell) \right] \frac{d\Gamma_{K*}}{dq^2} + W_9 \end{aligned} \quad (43)$$

with

$$\begin{aligned} W_9 &= \frac{1}{2\pi} \left[2\tilde{J}_{1a}^c + 2\tilde{J}_{2a}^c \cos 2\theta_\ell + (\tilde{J}_{1b}^c + \tilde{J}_{2b}^c \cos 2\theta_\ell)(\cos \hat{\theta}_K - \sin \hat{\theta}_K) \right. \\ &\quad \left. + \cos \hat{\phi} (\tilde{J}_5 + 2\tilde{J}_4 \cos \theta_\ell)(\cos \hat{\theta}_K + \sin \hat{\theta}_K) \sin \theta_\ell \right] \end{aligned} \quad (44)$$

Finally, notice that the list is not exhaustive and other foldings are also possible.

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